Exercise Set Solutions #7
"Discrete Mathematics" (2025)

E1. Compute the values for $\mu(10!), \phi(10!)$, and $\mu(2025)$.

> ***Solution:*** Since $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, there is at least one prime power with exponent strictly larger than 1 and therefore $\mu(10!) = 0$. Due to the prime factorization of 10 !, we get $\phi(10!) = 10!\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right) = 829440$.
> The prime factorization of 2021 is $3^4 \times 5^2$, therefore $\mu(2025) = 0$.

E2. Show that $n = \sum_{d|n} \phi(d)$ and that $\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$.

> ***Solution:*** Let $p_1, p_2, \ldots, p_k \in \mathbb{Z}_{\geq 1}$ be all the prime numbers dividing n. Then we know that
>
> $$\begin{aligned} \phi(n) &= n \prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right) \\ &= n\left(1-\sum_{i}\frac{1}{p_i}+\sum_{1\leq i<j\leq k}\frac{1}{p_i p_j}-\sum_{1\leq i<j<k\leq l}\frac{1}{p_i p_j p_k}+\ldots\right). \end{aligned}$$
>
> The above sum is actually $\sum_{d|n} \frac{\mu(d)}{d}$. Indeed, $\mu(d)$ is zero if $p_i^2 \mid d$ for some i so the only terms that appear are when d is a product of a subset of primes in $p_1, p_2, \ldots, p_k$. The second identity then follows by applying the Mobius inversion formula with $f(n) = n$ and $g(n) = \phi(n)$.

E3. Let $\sigma(n)$ denote the sum of all positive divisors of a number n. For instance, $\sigma(6) = 1+2+3+6 = 12$. Prove that $\phi(n) + \sigma(n) \geq 2n$, for all $n \in \mathbb{N}$, and characterize all n such that equality is achieved.

> ***Solution:*** Note that
>
> $$\sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} = n \sum_{d|n} \frac{1}{d}$$
>
> Using Exercise 2 above, we get that
>
> $$\phi(n) + \sigma(n) = n \sum_{d|n} \frac{\mu(d)+1}{d}$$
>
> Since $\mu(m) + 1 \geq 1$ for each $m \in \mathbb{N}$, and since $\mu(1) = 1$, we conclude the desired inequality. For the case of equality, note that if n has at least two distinct prime factors p_1, p_2, then $\mu(p_1 p_2) = 1$, and hence we obtain a non-trivial factor in the sum before. Hence, $n = p^k$ for a prime p is necessary in order for equality to hold. On the other hand, since $\mu\left(p^2\right) = 0$, we get non-trivial factors whenever $k \geq 2$, and hence equality holds if and only if $n = p$, where p is prime.

E4. Let $\Lambda(n)$ be a function defined for $n \in \mathbb{Z}_{\geq 1}$ by the rule

$$\sum_{d|n} \Lambda(d) = \log n$$

Show that

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime number } p \text{ and } k \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

> We prove it by induction in the number of prime divisors of n (counting multiplicty). For $n = p^k$ the statement follows easily. Let us assume that the statement is true for $m = q_1^{b_1} \cdots q_r^{b_l}$ with $b_i \neq 0$ for all i and $\sum_{i=1}^{l} b_i = K$. Assume that $n = p_1^{a_1} \cdots p_r^{a_r}$ with $a_i \neq 0$ for all i and $\sum_{i=1}^{r} a_i = K + 1$. Then
>
> $$\log(n) = \sum_{d|n} \Lambda(d) = \sum_{d|p_1^{a_1} \cdots p_{r-1}^{a_{r-1}}} \Lambda(d) + \sum_{d|p_1^{a_1} \cdots p_r^{a_r - 1}} \Lambda(dp_r).$$
>
> Thus, we have that
>
> $$\sum_{d|p_1^{a_1} \cdots p_r^{a_r - 1}} \Lambda(dp_r) = \log(n) - \log(p_1^{a_1} \cdots p_{r-1}^{a_{r-1}}) = \log(p_r^{a_r}).$$
>
> By induction hypothesis, we have that
>
> $$\sum_{d|p_1^{a_1} \cdots p_r^{a_r - 1}} \Lambda(dp_r) = \Lambda(n) + a_r \Lambda(p_r).$$
>
> Thus, using the base case, we get that $\Lambda(n) = 0$, concluding.
> **Note:** The problem can also be solved by using Möbius inversion formula with the formula defining Λ, and then developing what one gets from it.

E5. Let $f, g : \mathbb{N} \to \mathbb{C}$ be two functions. We define their Dirichlet product to be

$$f * g(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

(a) Let the Dirichlet series associated with f, g be defined, respectively, as

$$F(s) = \sum_{n \geq 1} \frac{f(n)}{n^s}, G(s) = \sum_{n \geq 1} \frac{g(n)}{n^s}$$

Suppose the sums defining F and G converge absolutely for all $s > s_0$. Prove that the series

$$H(s) = \sum_{n \geq 1} \frac{f * g(n)}{n^s}$$

converges absolutely for $s > s_0$ as well, and prove that

$$H(s) = F(s)G(s)$$

(b) Let

$$\zeta(s) = \sum_{n \geq 1} n^{-s}$$

denote the Riemann Zeta function. Conclude that

$$\frac{1}{\zeta(s)} = \sum_{n \geq 1} \frac{\mu(n)}{n^s}$$

for all $s > 1$.

Solution: (a) Let us show absolute convergence of H first: we have

$$\sum_{n\geq 1} \frac{|f*g(n)|}{n^s} \leq \sum_{n\geq 1} \left(\sum_{d|n} |f(d)| \left| g\left(\frac{n}{d}\right) \right| \right) n^{-s}$$
$$= \sum_{d\geq 1} |f(d)| \left(\sum_{k\geq 0} |g(k)| (dk)^{-s} \right)$$
$$= \left(\sum_{d\geq 1} |f(d)| d^{-s} \right) \left(\sum_{k\geq 1} |g(k)| k^{-s} \right).$$

Since we know that both of the series above in the last equality converge absolutely for $s > s_0$, so does the original one. In order to prove the desired identity for H, one simply observes that

$$\sum_{n\geq 1} \frac{f*g(n)}{n^s} = \sum_{n\geq 1} \left(\sum_{dk=n} f(d)g(k) \right) n^{-s}$$
$$= \sum_{d\geq 1} f(d) \left(\sum_{k\geq 0} g(k)(dk)^{-s} \right)$$
$$= \left(\sum_{d\geq 1} f(d) d^{-s} \right) \left(\sum_{k\geq 1} g(k) k^{-s} \right) = F(s)G(s)$$

(b) Let us use part (a): since, by Möbius inversion, we have $\mu * 1(n) = 1$ if $n = 1$ and 0 otherwise, by part (a) we have

$$1 = \sum_{n\geq 1} \frac{\mu * 1(n)}{n^s} = \zeta(s) \left(\sum_{k\geq 1} \frac{\mu(k)}{k^s} \right)$$

whenever $s > 1$ - here, we used the (not too hard to verify directly) fact that part (a) holds with $s_0 = 1$. This finishes the problem.